

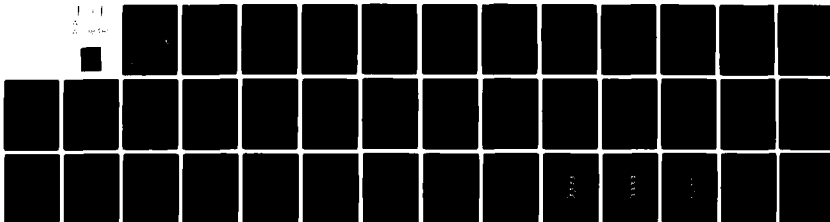
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RATIONAL SPECTRAL ESTIMATION: A HIGH PERFORMANCE-FAST ALGORITHM--ETC(U)
APR 81 J A CADZOW N00014-80-C-0303

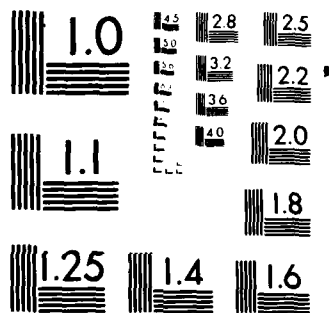
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The new method is predicated on selecting the parameters of a rational spectral model so that the underlying Yule-Walker equations are best approximated over an extended time span. This approach was incorporated by Cadzow in evolving an effective pole-zero spectral model (e.g., see refs. [1],[2],[22]). More recently, Ogura and Yoshida have adopted this same procedure for obtaining a more specialized all pole spectral model [23].

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6 RATIONAL SPECTRAL ESTIMATION: A HIGH
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ABSTRACT

It is widely recognized that spectral estimation serves as a powerful digital signal processing tool in such diverse areas as radar, array processing, adaptive filtering, seismology, and, speech processing. In this paper, a novel method of rational spectral estimation shall be presented which possesses a number of admirable properties: (i) it has an elegant algebraic structure, (ii) its spectral estimation performance has been empirically found to be superior to such contemporary methods as the maximum entropy and Box-Jenkins methods, and (iii) it is implementable by a "fast algorithm" which is computationally competitive with recently developed fast LMS algorithms. Taken in combination, these properties mark this new method as being a primary spectral estimation tool in those challenging applications requiring high performance spectral estimation in a real time setting.

The new method is predicated on selecting the parameters of a rational spectral model so that the underlying Yule-Walker equations are best approximated over an extended time span. This approach was incorporated by Cadzow in evolving an effective pole-zero spectral model (e.g., see refs. [1],[2],[22]). More recently, Ogura and Yoshida have adopted this same procedure for obtaining a more specialized all pole spectral model [23].

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I. INTRODUCTION

In many signal processing applications, it is necessary to estimate the power spectral density which characterizes a wide-sense stationary time series $\{x(n)\}$. This estimate is to be based upon a finite set of time series observations which is here taken to be the n contiguous measurements

$$x_1, x_2, \dots, x_n \quad (1)$$

A standard procedure for obtaining a spectral estimate is to first postulate a parametric spectral model, and, then to use the above time series observations for fixing the model's parameters. In light of this given observation set, it is clear that the spectral model used must be dependent on only a finite set of parameters. Without doubt, the most widely employed model is the so-called rational spectral density model as specified by

$$S_x(\omega) = \frac{|b_0 + b_1 e^{-j\omega} + \dots + b_q e^{-jq\omega}|^2}{|1 + a_1 e^{-j\omega} + \dots + a_p e^{-jp\omega}|^2} \quad (2)$$

This spectral density model is generally referred to as an autoregressive-moving average (ARMA) model of order (p,q) . The importance ascribed to this model arises from the fact that any continuous spectral density can be approximated arbitrarily closely by a rational function of form (2) by proper choice of the order (p,q) and the governing a_k, b_k coefficients [3].

Although this ARMA model is the most general rational spectral density model, the more specialized autoregressive (AR) model for which $q = 0$, and, the moving average (MA) model for which $p = 0$ have received the preponderance of attention from researchers and users alike. For instance, the maximum entropy, one-step predictor, and, autoregressive methods have been developed for efficiently estimating an AR model's a_k

coefficients. Similarly, the periodogram approach and its variants have been developed for efficiently estimating a MA model's b_k coefficients. The interested reader will find excellent treatments of these and other rational spectral estimation methods in Haykin [4] and Childers [5].

The primary reasons for concentrating on AR and MA spectral models have been that: (i) they have proven effective in achieving satisfactory spectral estimation performance, and, (ii) these particular modeling approaches result in tractable analyses and efficient implementations. It is widely recognized, however, that an ARMA spectral model will generally lead to an improved performance in which fewer model parameters are used. In recognition of this fact, a variety of procedures have been developed for generating ARMA models. These include the whitening filter approach which is typically iterative in nature, generally slow in convergence, and, usually requires a rather large number of time series observations to be effective (e.g., see refs. [6] and [7]). More desirable closed form procedures which overcome most of these deficiencies have been proposed. These include the so-called Box-Jenkins method and its variants [8]-[10], and, more recently, Cadzow has developed the "high performance" method [1] and [2]. Although this latter method has provided excellent spectral estimation performance when compared to its ARMA and AR model counterparts, its computational efficiency is relatively inferior.

Recently, attention has been directed towards developing "fast" spectral estimation algorithms. These algorithms are typically based on the divide and conquer approach. As an example, it is possible to use this approach for estimating the autoregressive coefficients of a p th order AR model with the number of required additions and multiplications

being on the order of $p \log(p)$ (e.g., see refs. [11]-[13]). These fast algorithms offer the exciting possibility of providing an adaptive spectral estimator in which the spectral model's parameters are updated as new time series observations become available. Unfortunately, the implementation of these fast algorithms tend to be rather complex and a relatively large value for the model order parameter p is required before the computational complexity $p \log(p)$ is approached. It is believed that future developments will alleviate these shortcomings.

With these thoughts in mind, it is apparent that there is a pressing need for a high performance ARMA spectral estimation method which will possess the computational efficiency of the fast algorithms. In this paper, a novel ARMA modeling procedure for achieving these two objectives will be presented. The primary objective of this paper will be that of developing the basic description of this method and illustrating its performance behavior by means of a number of comparative examples. The details of how one may implement this algorithm so as to attain the aforementioned fast computational complexity will be given in a subsequent paper [14].

We shall herein consider primarily the task of estimating the ARMA model's a_k autoregressive coefficients. This estimation will be, to a large extent, motivated by the well known Yule-Walker equations which characterize stationary ARMA time series. The Yule-Walker equation development and other fundamental concepts will now be briefly reviewed.

II. PRELIMINARY CONCEPTS

In this section, a number of concepts which are central to the development of this paper's ARMA model spectral estimation method are presented. Undoubtedly the key notion is contained in the Yule-Walker equations which characterize rational stationary time-series.

Yule-Walker Equations

It is readily shown that the stationary random time series whose spectral density is specified by relationship (2) can be modeled as being the response of the causal ARMA system

$$x_k + \sum_{i=1}^p a_i x_{k-i} = \sum_{i=0}^q b_i \varepsilon_{k-i} \quad (3)$$

to a zero mean white noise excitation $\{\varepsilon_n\}$ whose individual terms have variance one. The autocorrelation description of this system is readily obtained by first multiplying each side of expression (3) by the entity x_{k-m}^* and then taking the expected value. This results in the well known Yule-Walker equations as specified by

$$r_x(m) + \sum_{i=1}^p a_i r_x(m-i) = 0 \quad \text{for } m > q \quad (4)$$

In this expression, the entries $r_x(m)$ denote the time series' autocorrelation elements as given by

$$r_x(m) = E\{x_k x_{k-m}^*\} \quad (5)$$

in which $*$ and E denote the operations of complex conjugation and expected value, respectively.

In what is to follow, the characteristic Yule-Walker equations (4) will serve as the basis for estimating the ARMA model's a_k and b_k coefficients from the given set of time series observations (1). The method to be described has the appealing property of being both computationally

efficient to implement, and, adaptive in nature. Namely, as the new time series element x_{n+1} becomes available, it will be possible to efficiently update the ARMA model's optimal autoregressive coefficients corresponding to the n data length set so as to obtain the optimal autoregressive coefficients for the new $n+1$ data length set. This capability is essential if real time spectral estimation is to be achieved.

Down and Up-Shift Operators

In the spectral estimation method to be presented, extensive utilization of the down shift operator S will be made. This operator is formally defined by

$$S\underline{x} = [0, x_1, x_2, \dots, x_{n-1}]' \quad (6)$$

in which \underline{x} is an $n \times 1$ vector as given by

$$\underline{x} = [x_1, x_2, \dots, x_n]' \quad (7)$$

where the prime symbol denotes vector transposition. As its name implies, operator S downshifts by one unit the elements of the vector upon which it operates and inserts a zero into the vacated first component position.

The downshift operator has the following $n \times n$ matrix representation

$$S = \begin{bmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \cdot \\ & & & & & 1 & 0 \end{bmatrix} \quad (8)$$

When this operator is applied sequentially m times to the vector \underline{x} , it is clear that a down-shift of m units will arise as follows

$$S^m \underline{x} = [\underbrace{0, 0, \dots, 0}_m, x_1, x_2, \dots, x_{n-m}]' \quad (9)$$

where the integer m is taken to be nonnegative.

In a similar fashion, the up-shift operator is formally defined by

$$S^{-1} \underline{x} = [x_2, x_3, \dots, x_n, 0]' \quad (10)$$

will also play a key role. We have here used an imprecise notation for the up-shift operator in the sense that S^{-1} does not designate the inverse of operator S . As a matter of fact, the inverse of S does not exist. The primary reason for using the notation S^{-1} will be due to the notational simplification which thereby arises. Namely, the operator S^m will serve the dual role of corresponding to a m down-shift operator when the integer $m \geq 0$ while corresponding to an up-shift operator whenever $m < 0$. In using the operators S and S^{-1} , it is important to appreciate the fact that

$$S^m S^n = S^{m+n} \quad \text{iff } m \text{ \& } n \geq 0 \text{ or } m \text{ \& } n < 0 \quad (11)$$

but that when the integers m and n are of opposite sign, this equality will be invalid. As a final comment it is a simple matter to show that the $n \times n$ matrix S^{-1} is specified by

$$S^{-1} = S' \quad (12)$$

that is, the up-shift operator is simply the matrix transpose of the down-shift operator.

Displacement Rank

The ability to achieve a fast algorithmic spectral estimator will be dependent upon one taking advantage of the near Toeplitz structure of certain matrices which arise in the procedures to be described. In

particular, the degree to which the $p \times p$ matrix A is Toeplitz in structure is measured by its "displacement rank" [15]. The displacement rank of matrix A is formally given by

$$\alpha(A) = \min[\alpha_-(A), \alpha_+(A)] \quad (13)$$

where

$$\alpha_-(A) = \text{rank}[A - S A S'] \quad (14)$$

$$\alpha_+(A) = \text{rank}[A - S' A S] \quad (15)$$

It is a simple matter to show that the displacement rank of a Toeplitz matrix is less than or equal to two. Thus a matrix whose displacement rank is near two is said to be close to Toeplitz in structure.

In solving the system of p linear equations in p unknowns as specified by

$$Ax = y$$

one may use standard matrix inversion methods to arrive at the solution. These routines will take on the order of p^3 (i.e., $\phi(p^3)$) multiplication and addition operations to obtain that solution. If the displacement rank of matrix A is α , however, one may use a generalized version of the Levinson algorithm to obtain the solution using $\phi(\alpha p^2)$ computations [16]. If α is sufficiently smaller than p , it then follows that a significant computational savings can be realized by adopting the generalized Levinson algorithm solution procedure. In what is to follow, use will be made of this observation.

III. ARMA SPECTRAL ESTIMATION

In this section, a novel procedure for estimating an ARMA model's autoregressive coefficients shall be presented. This development is begun by first evaluating the model equation (3) over the integer set $p+1 \leq k \leq n$ to obtain the time series relationships

$$\begin{bmatrix} x_{p+1} \\ x_{p+2} \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} x_p & x_{p-1} & \cdots & x_1 \\ x_{p+1} & x_p & \cdots & x_2 \\ \vdots & \vdots & & \vdots \\ x_{n-1} & x_{n-2} & \cdots & x_{n-p} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} \epsilon_{p+1} & \epsilon_p & \cdots & \epsilon_{p-q+1} \\ \epsilon_{p+2} & \epsilon_{p+1} & \cdots & \epsilon_{p-q+2} \\ \vdots & \vdots & & \vdots \\ \epsilon_n & \epsilon_{n-1} & \cdots & \epsilon_{n-q} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_q \end{bmatrix} \quad (16a)$$

In what is to follow, it will be convenient to represent these relationships in the compact vector space format

$$\underline{x} + X \underline{a} = \epsilon \underline{b} \quad (16b)$$

where \underline{x} , \underline{a} and \underline{b} are $n-p$, p , and $q+1$ column vectors, respectively, while X and ϵ are $(n-p) \times p$ and $(n-p) \times (q+1)$ "Toeplitz type" matrices, respectively. The entries of these individual vectors and matrices are directly obtained upon examination of relationships (16a) and (16b).

It is now desired to utilize relationship (16b) in conjunction with the Yule-Walker equations (4) to effect a procedure for estimating the ARMA model's autoregressive coefficients. As we will see, this objective is attained by first introducing the following $(n-p) \times t$ Toeplitz type matrix

$$Y = \begin{bmatrix} x_{p-q} & x_{p-q-1} & \cdots & x_{p-q-t+1} \\ x_{p-q+1} & x_{p-q} & \cdots & x_{p-q-t+2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-q-1} & x_{n-q-2} & \cdots & x_{n-q-t} \end{bmatrix} \quad (17)$$

in which the convention is adopted of setting to zero any matrix entry x_k for which k lies outside the observation index interval.¹ The integer t , which specifies the number of columns of matrix Y , will be found to correspond to the number of Yule-Walker equations that are being approximated (i.e., relationship (4) for $q < m \leq q+t$). A discussion on how one selects the value of t will be shortly given.

To achieve the desired Yule-Walker equation approximation, let us now premultiply each side of relationship (16b) by the complex conjugate transpose of matrix Y as denoted by Y^\dagger to yield

$$Y^\dagger \underline{x} + Y^\dagger X \underline{a} = Y^\dagger \underline{e} \underline{b} \quad (18)$$

Upon taking the expected value of this system of t equations, it is found that for the case when the ARMA model's numerator order is greater than or equal to its denominator order (i.e., $q \geq p$), this yields

$$(n-m) \left\{ r_x(m) + \sum_{k=1}^p a_k r_x(m-k) \right\} = 0 \quad q < m \leq q+t \quad (19a)$$

while for the case $q < p$, we find that

$$(n-p) \left\{ r_x(m) + \sum_{k=1}^p a_k r_x(m-k) \right\} = 0 \quad q < m \leq p \quad (19b)$$

$$(n-m) \left\{ r_x(m) + \sum_{k=1}^p a_k r_x(m-k) \right\} = 0 \quad p < m \leq q+t$$

¹A more generalized version of this modeling process may be obtained upon substituting the integer s for p wherever it appears in relationship (16) and (17). For simplicity of presentation, the special case $s = p$ is herein considered.

In either order case, it is therefore seen that the system of equations (18) provides an "unbiased estimate of the first t Yule-Walker equations. It is important to note that the right side vector $Y^T \underline{e}_b$ has an expected value of $\underline{\theta}$ (the $t \times 1$ zero vector). This is a direct consequence of the ARMA model's causality and the whiteness of the excitation process which renders $E\{x_m \epsilon_k^*\} \equiv 0$ for all $m < k$.

With these thoughts in mind, an appealing procedure for selecting the ARMA model's p autoregressive coefficients is suggested. Namely, they will be selected in a manner so as to best "satisfy" the system of linear equations

$$Y^T \underline{x} + Y^T X_a = \underline{\theta} \quad (20)$$

which constitutes the aforementioned unbiased estimate of the first t Yule-Walker equations. In a real sense, this selection process constitutes a procedure which is "most" consistent with the given time series observations and the hypothesized ARMA model of order (p, q) . The elements of the vector $Y^T \underline{x}$ and matrix $Y^T X$ for this "unmodified" method are given in Table 1.

Relationship (20) represents a system of t linear equations in the p autoregressive coefficient unknowns. As such, a "best" solution to this system of equations is meaningful only in those cases whereby $t \geq p$. In what is to follow, it will be then assumed that $t \geq p$, and, that the rank of the $t \times p$ matrix $Y^T X$ is p . Under these constraints, it is clear that the system of equations (20) will be consistent when $t = p$ and will be generally inconsistent when $p < t \leq n - q - 1$. The upper limit on t (i.e., $n - q - 1$) constitutes the largest Yule-Walker equation approximation which is consistent with the given ARMA model and the data of length n . With this in mind, let us now separately treat the cases $t = p$ and $p < t \leq n - q - 1$.

Case 1 : $t = p$

In this case, the $p \times p$ system of linear equations (20) has a unique solution. If standard matrix inversion techniques were used to obtain this solution, the required number of multiplication and addition computations would be on the order of p^3 (i.e. $\phi(p^3)$). It is possible, however, to take advantage of the near Toeplitz structure of the $p \times p$ matrix Y^+X to achieve a more efficient solution procedure. Specifically, it is readily shown that the displacement rank of this matrix is less than or equal to 4. This being the case, one may then use a generalized Levinson algorithm to obtain the solution to relationship (20) using on the order of $4p^2$ computations. Thus, the autoregressive coefficient selection procedure as given by equation (20) is computationally competitive with the maximum entropy method. The spectral estimation performance of this ARMA model procedure, however, typically provides substantial improvement over the maximum entropy method.

Relationship (20) bears some resemblance to the first iterate of the Box-Jenkins method [8] of ARMA spectral estimation in the sense that the first p Yule-Walker equations (i.e. equation (4) with $q < m \leq q+p$) are being approximated. Upon closer examination, however, it is found that the standard Box-Jenkins method does not possess the algebraic structure which enables one to utilize sophisticated LMS concepts to effect a super efficient solution procedure for solving (20). As a final note, it is a simple matter to show for the special case $q = 0$ and $t = p$, that matrix $Y = X$ and relationship (20) reduces to the well-known covariance method for obtaining an AR spectral model (e.g., see refs. [17] and [18]).

Case 2 : $p < t \leq n-q-1$

When $t > p$, the system of t linear equations (20) in the p autoregressive coefficients is overdetermined. As such, it will not be generally possible to find a solution to expression (20). It is possible, however, to select an autoregressive coefficient vector \underline{a} which causes this system of equations to be best approximated in the least squared error sense. This entails the introduction of the following quadratic functional

$$f(\underline{a}) = [\underline{Y}^T \underline{x} + \underline{Y}^T \underline{X} \underline{a}]^T \Lambda [\underline{Y}^T \underline{x} + \underline{Y}^T \underline{X} \underline{a}]$$

which measures the degree to which the system of linear equations (20) is being approximated. The $t \times t$ diagonal matrix Λ is taken to be positive-semidefinite.² It is a simple matter to show that the autoregressive coefficient vector which minimizes this quadratic function satisfies the normal equations

$$\underline{X}^T \underline{Y} \Lambda \underline{Y}^T \underline{X} \underline{a} = - \underline{X}^T \underline{Y} \Lambda \underline{Y}^T \underline{x} \quad (21)$$

A solution to this linear system of p equations then provides the autoregressive coefficient estimates for the postulated ARMA model. It is of interest to note that this system of linear equations reduces to the Cadzow High Performance method of ARMA spectral estimation for the particular selection $q = p$ (e.g., see refs. [1] & [2]).

The improved spectral estimation performance achieved in using the estimation method (21) over such competitors as the Box-Jenkins method is a

²The diagonal elements λ_{kk} of the matrix Λ are typically selected to be nonincreasing (i.e., $\lambda_{kk} \geq \lambda_{k+1k+1}$) so as to reflect an anticipated increase in the variance of the Yule-Walker equation approximations for increasing k as evident from relationships (19). Choices of $\lambda_{kk} = (n-q-k)^4$ or $\lambda_{kk} = (1)^k$ have been found to yield satisfactory performance.

direct consequence of selecting integer t to be larger than the minimal value p . With the corresponding larger set of Yule-Walker equations that are being approximated, it intuitively follows that the modeling autoregressive coefficients will be less sensitive to inaccuracies of the correlation estimates as embodied in Y^+X and $Y^+\underline{x}$ than in the minimal value situation when $t = p$. This anticipation of a spectral estimation performance improvement has in fact been demonstrated on a rather large number of numerical examples carried out by the authors and others.

From a computational viewpoint, the primary advantage accrued in using the herein developed autoregressive coefficient estimation procedure resides in the specific algebraic structure thereby obtained. This structure may be used to effect particular "fast" algorithmic solution procedures for the autoregressive coefficient estimates. These fast algorithms are computationally much more efficient than the aforementioned generalized Levinson algorithm. In order to achieve the required structure for these fast algorithms, it will be necessary to make minor modifications on the constituent vector \underline{x} and matrices X and Y . In the next section, a discussion of some candidate modifications will be made.

IV. PRE AND POSTMODIFICATION METHODS

It is possible to realize significant computational savings in the proposed ARMA spectral estimation method over that provided by the generalized Levinson algorithm. This improvement will entail a slight modification in the constituent vector \underline{x} and matrices X and Y which constitute the Yule-Walker equation estimations (20). Although the suggested modifications will typically result in biased estimates of the Yule-Walker equations, it is shown that when the data length n adequately exceeds the order parameters p and q (i.e., $n \gg p \& q$) that these estimates are virtually unbiased.

In addition to these vector and matrix modifications, it will be necessary to restrict the integer t to its minimal value of p in order to achieve the fast algorithm solution capability. Unfortunately, the restriction $t = p$ will generally result in an associated decrease in spectral estimation performance. Thus, in obtaining a computationally fast algorithmic solution procedure for the a_k coefficients, an accompanying sacrifice in spectral estimation performance is the price being paid. One must therefore carefully consider the ramifications of this tradeoff for any given application. Fortunately, the degradation in performance is not great for many relevant applications.

With these remarks in mind, we shall now consider the aforementioned modifications in the constituent \underline{x} , X , and Y entries. Although a countless number of modifications are possible, we will be concerned exclusively with the so-called "premodification" and "postmodification" procedures. Additional modification procedures are now being examined and will subsequently be reported upon.

(i) Premodification Method

In the premodification method, the vector \underline{x} , and, matrices X and Y used in expressions (16) and (17) are modified as follows. The vector \underline{x} becomes the given observed time-series vector

$$\underline{x}_1 = [x_1, x_2, \dots, x_n]' \quad (22a)$$

while the $n \times p$ matrix X has the p column vector structure

$$X_1 = [\underline{Sx}_1 \quad \vdots \quad S^2 \underline{x}_1 \quad \vdots \quad \dots \quad \vdots \quad S^p \underline{x}_1] \quad (22b)$$

and the $n \times t$ matrix Y has the t column vector structure

$$Y_1 = [S^{q+1} \underline{x}_1 \quad \vdots \quad S^{q+2} \underline{x}_1 \quad \vdots \quad \dots \quad \vdots \quad S^{q+t} \underline{x}_1] \quad (22c)$$

where S is the aforementioned $n \times n$ down-shift matrix. When these entries are substituted into relationship (20), the so-called premodification method of estimating the ARMA model's autoregressive coefficients is at hand. This method is suggestively referred to as being premodified since, among other things, the original system of equations (16) has appended to it a set of p new equations as represented by the first p rows of matrix X_1 (the last $n-p$ rows of matrix X_1 are equal to matrix X). These first p rows form a lower triangular matrix and, as such, an undesirable "transient" effect on the ARMA modeling is introduced. It is precisely because of this triangular structure, however, that we are able to obtain a fast computational algorithm for achieving estimates of the ARMA models' autoregressive coefficients[14].

If the modifications as embodied in expressions (22) are substituted into relationship (20), the following new approximation of the first t Yule-Walker equations is obtained

$$Y_1^+ X_1 \underline{a} + Y_1^+ \underline{x}_1 = 0 \quad (23)$$

Upon taking the expected value of this relationship, it is found that for the typical ARMA model order selection case $q \leq p$, the following biased estimates of the Yule-Walker equations result

$$(n-m) \sum_{k=0}^m a_k r_x(m-k) + \sum_{k=m+1}^p (n-k) a_k r_x(m-k) = 0 \quad q < m < p \quad (24a)$$

$$(n-m) \sum_{k=0}^p a_k r_x(m-k) = 0 \quad p \leq m \leq q+t \quad (24b)$$

in which $a_0 = 1$. It is to be noted that these estimates are virtually unbiased whenever $n \gg p$. In a similar manner, for the case $q > p$, the expectation operation results in the unbiased results

$$(n-m) \sum_{k=0}^p a_k r_x(m-k) = 0 \quad q < m \leq q+t \quad (24c)$$

Thus, in any case, the linear equation (23) provides a suitably good approximation to the underlying Yule-Walker equations.

Special Case : $t = p$

The primary advantage accrued by introducing the premodified modification is computational in nature. It is a simple matter to show that, in the special case $t = p$, the $p \times p$ matrix $Y_1^+ X_1$ has a displacement rank less than or equal to 3. As such, a generalized Levinson algorithm may be developed for solving relationship (23) which requires $\phi(3p^2)$ computations. More importantly, however, is the fact that the particular structure of matrices X_1 and Y_1 will enable one to utilize a fast algorithmic procedure to obtain this solution using $\phi(p)$ computations. This algorithm is described in reference [14].

(ii) Postmodification Method

When employing the postmodification method, the column vector \underline{x} is replaced by the $n \times 1$ vector

$$\underline{x}_2 = S^{-p} \underline{x}_1 \quad (25a)$$

where \underline{x}_1 is given by relationship (22a). The matrix X is replaced by the $n \times p$ matrix

$$X_2 = [S^{-p+1} \underline{x}_1 : S^{-p+2} \underline{x}_1 : \dots : \underline{x}_1] \quad (25b)$$

while the matrix Y is replaced by the $n \times t$ matrix

$$Y_2 = [S^{-p+q+1} \underline{x}_1 : S^{-p+q+2} \underline{x}_1 : \dots : S^{-p+q+t} \underline{x}_1] \quad (25c)$$

Upon substitution of these entries into expression (20), the so-called postmodification method of estimating the ARMA model's autoregressive coefficients is obtained. This method is referred to as being postmodified since, among other things, the original system of equations (16) are here being supplemented with p new equations as represented by the last p rows of matrix X_2 (the first $n-p$ rows of matrix X_2 are equal to matrix X). These last p rows form a lower triangular matrix and introduce an undesirable transient effect on the resultant autoregressive coefficient solution. The primary advantage obtained in using the postmodification method is computational in nature as will now be briefly described.

With these postmodified matrix and vector entries, the following approximation to the first t Yule-Walker equations is obtained

$$Y_2^+ X_2 \underline{a} + Y_2^+ \underline{x}_2 = \underline{\theta} \quad (26)$$

The degree to which this linear system of equations approximates the Yule-Walker equation may be measured by taking its expected value. This is found to yield for the more typical order selection case $q \leq p$

$$\sum_{k=0}^m (n-p+k) a_k r_x(m-k) + (n-p+m) \sum_{k=m+1}^p a_k r_x(m-k) = 0 \quad q < m \leq p \quad (27a)$$

$$\sum_{k=0}^p (n-m+k) a_k r_x(m-k) = 0 \quad p < m \leq q+t \quad (27b)$$

while for the less typical case $q > p$, one finds that

$$\sum_{k=0}^p (n-m+k)a_k r_x(m-k) = 0 \quad q < m \leq q+t \quad (27c)$$

Thus, relationship (26) represents a biased estimate of the Yule-Walker equations. It is to be noted, however, that when the number of time series observations n adequately exceeds the ARMA model parameter p , relationship (26) provides an almost unbiased estimate.

Special Case : $t = p$

When $t = p$, it is a simple matter to show that the displacement rank of the $p \times p$ matrix $Y_2^+ X_2$ is less than or equal to three. A generalized Levinson algorithm may therefore be used to solve relationship (26) which requires $\phi(3p^2)$ computations. It is possible, however, to utilize the algebraic structure of relationship (26) to generate a fast algorithmic solution procedure that requires $\phi(p \log p)$ computations as presented in reference [14].

(iii) Pre-and-Postmodification

If the pre and postmodification procedures are simultaneously incorporated, a third procedure arises. Specifically, the column vector \underline{x} is replaced by the $(n+p) \times 1$ vector

$$\underline{x}_3 = [\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n, \underbrace{0, \dots, 0}_{p \text{ zeroes}}]' \quad (28a)$$

The matrix X is next replaced by the $(n+p) \times p$ matrix

$$X_3 = [\underline{s}\underline{x}_3 \vdots \underline{s}^2 \underline{x}_3 \vdots \dots \vdots \underline{s}^p \underline{x}_3] \quad (28b)$$

while the matrix Y is replaced by the $(n+p) \times t$ matrix

$$Y_3 = [\underline{s}^{q+1} \underline{x}_3 \vdots \underline{s}^{q+2} \underline{x}_3 \vdots \dots \vdots \underline{s}^{q+t} \underline{x}_3] \quad (28c)$$

If these entries are substituted into relationship (20), the so-called "Pre-and-Postmodification" method of ARMA model autoregressive co-efficient estimation procedure is generated. In using this particular modification, it is readily shown that the original system of equations (16) are being supplemented with $2p$ new equations as represented by the first and last p rows of matrix X_3 (the centermost $n-p$ rows of matrix X_3 are equal to matrix X). The initial and final p rows introduce an associated deleterious transient effect. The resultant algebraic structure for the system of linear equations (11), however, result in a significant computational savings in determining the autoregressive coefficients.

In using the Pre-and-Postmodification method, the approximation to the first t Yule-Walker equations is provided by the linear system of equations

$$Y_3^+ X_3 \underline{a} + Y_3^+ \underline{x}_3 = \underline{\theta} \quad (29)$$

This system of equations provides a biased estimate of the Yule-Walker equations as is made apparent upon taking its expected value. For the typical ARMA modeling order selection of $q \leq p$, this expectation is found to yield

$$\sum_{k=0}^m (n-m+k) a_k r_x(m-k) + \sum_{k=m+1}^p (n+m-k) a_k r_x(m-k) = 0 \quad q < m < p \quad (30a)$$

$$\sum_{k=0}^p (n-m+k) a_k r_x(m-k) = 0 \quad p < m \leq q+t \quad (30b)$$

while for the selection $q > p$, we have

$$\sum_{k=0}^p (n-m+k) a_k r_x(m-k) = 0 \quad q < m < q+t \quad (30c)$$

Although the biased nature of estimate (29) is revealed by expressions (30), it is clear that when the number of time series observations n adequately exceed the ARMA model order parameter p , the estimate becomes almost unbiased.

Special Case : $t = p$

In this case, the displacement rank of the $p \times p$ matrix $Y_3^+ X_3$ is found to be less than or equal to two. A generalized Levinson algorithm may then be utilized to obtain the solution to expression (29) using $\phi(2p^2)$ computations. As in the other modified methods, however, one may utilize even more efficient solution procedures to obtain this solution using $\phi(p \log p)$ computations [14].

The following Table is useful when programming the autoregressive coefficient selection procedure as embodied in expressions (20 or (21).

Method	$A = Y^+X$ a_{ij} for $1 \leq i \leq t$ $1 \leq j \leq p$	$\underline{b} = Y^+\underline{x}$ b_i for $1 \leq i \leq t$
Unmodified	$a_{ij} = \sum_{k=1}^{n-p} x_{p+k-j} x_{p-q+k-i}^*$	$b_i = \sum_{k=1}^{n-p} x_{p+k} x_{p-q+k-i}^*$
Premodified	$a_{ij} = \sum_{k=1}^n x_{k-j} x_{k-q-i}^*$	$b_i = \sum_{k=1}^n x_k x_{k-q-i}^*$
Postmodified	$a_{ij} = \sum_{k=1}^n x_{p+k-j} x_{p-q+k-i}^*$	$b_i = \sum_{k=1}^n x_{p+k} x_{p-q+k-i}^*$
Pre & Postmodified	$a_{ij} = \sum_{k=1}^{n+p} x_{k-j} x_{k-q-i}^*$	$b_i = \sum_{k=1}^{n+p} x_k x_{k-q-i}^*$

TABLE 1: Entries of the matrix Y^+X and vector $Y^+\underline{x}$ for the various methods. The convention is here adopted of setting $x_k = 0$ whenever k is not in the observation index interval $1 \leq k \leq n$.

Method	Displacement Rank	Fast Algorithm Computational Measure
Unmodified	4	
Premodified	3	$\phi(p)$
Post modified	3	$\phi(p \log p)$
Pre & Postmodified	2	$\phi(p \log p)$

TABLE 2: Computational characterization of various methods for the minimal selection $t = p$.

V. MODEL ORDER SELECTION

An important consideration in any rational spectral estimation method is that of the determination of the model's order. A particularly attractive procedure exists for this objective relative to the method developed in this paper. It is predicated on the observation that the ARMA model's autoregressive coefficients are obtained by solving a system of p consistent linear equations in the p autoregressive coefficient unknowns. The form of these equations is

$$A \underline{a} = \underline{b} \quad (31)$$

in which A is an approximation of an autocorrelation matrix. This is exemplified by equation (20) with $t = p$ where $A = Y^T X$. If the time series being analyzed is ARMA of order \bar{p} , the matrix A conceptually becomes ill-conditioned when the selected model order p exceeds the time series order \bar{p} . Thus, the model order determination can be achieved by examining the conditioning of the matrix A as a function of p . As p is increased, the appropriate choice will be that value \hat{p} for which there is a precipitate decrease in conditioning when $p = \hat{p} + 1$. This approach, as applied to the high performance method of ARMA spectral estimation [1], was suggested by Pao and Lee [20].

There exists a variety of procedures for measuring the conditioning of a $p \times p$ matrix A . One of the more popular such measures is provided by the normalized determinant as defined by

$$C(A) = \det(A) / \sqrt{\sum_{i=1}^p \sum_{j=1}^p |a_{ij}|^2} \quad (32)$$

where $\det(A)$ denotes the determinant of matrix A . It is to be noted that this normalized determinant equals zero when the rank of the $p \times p$ matrix is less than p . This method has been found to provide an effective model order determinate [20].

VI. NUMERATOR DYNAMICS

A variety of procedures exist for determining the numerator dynamics of an ARMA time series once the AR coefficients have been estimated. In this section, a procedure which has been found to be particularly effective shall be presented. It makes use of the governing ARMA relationship (3) that models the underlying time series.

In this approach to estimating the numerator dynamics, we first introduce the so-called causal image of a time series' autocorrelation sequence as specified by

$$r_x^+(n) = -\frac{1}{2} r_x(0)\delta(n) + r_x(n)u(n) \quad (33)$$

in which $\delta(n)$ and $u(n)$ designate the standard unit-sample and unit-step sequences, respectively. Making use of the complex conjugate symmetrical property of stationary autocorrelation sequences, it then follows that the autocorrelation sequence can be uniquely recovered from its causal image according to the simple relationship

$$r_x^+(n) = r_x^+(n) + r_x^{+*}(-n) \quad (34)$$

Upon taking the discrete-Fourier transform of this relationship, it follows that the time series' spectral density is given by

$$\begin{aligned} S_x(\omega) &= S_x^+(\omega) + S_x^{+*}(\omega) \\ &= 2 \operatorname{Re} \{ S_x^+(\omega) \} \end{aligned} \quad (35)$$

where $S_x^+(\omega)$ designates the discrete-Fourier transform of the causal image sequence $\{r_x^+(n)\}$. According to this relationship, one may obtain the required spectral density estimate by alternatively estimating $S_x^+(\omega)$. This will be the approach now taken.

It is possible to obtain a convenient closed form expression for the function $S_x^+(\omega)$. This first entails introduction of an auxiliary sequence $\{c_k\}$ whose elements are specified by

$$c_m = r_x^+(m) + \sum_{k=1}^p a_k r_x^+(m-k) \quad (36)$$

After examination of the Yule-Walker equations (4) which govern the ARMA time series and noting the causality of the sequence $\{r_x^+(k)\}$, it follows that this auxiliary sequence is itself causal and finite in length. Namely,

$$c_m \equiv 0 \quad \text{for} \quad 0 > m > u = \max(q,p) \quad (37)$$

Upon taking the discrete Fourier transform of relationship (36) and incorporating identity (37), we have after rearrangement of terms

$$S_x^+(\omega) = \frac{c_0 + c_1 e^{-j\omega} + \dots + c_u e^{-ju\omega}}{1 + a_1 e^{-j\omega} + \dots + a_p e^{-jp\omega}} \quad (38)$$

where it is recalled that $u = \max(q,p)$. If this expression is then substituted into relationship (35), the required formulation of the spectral density is completed.

In summary, the procedure here presented for ascertaining the numerator dynamics effect on the overall spectral estimate entails the following four step procedure. One first computes estimates of the autocorrelation elements $r_x(0), r_x(1), \dots, r_x(u)$ from the given time series observations (1). One may use any procedure such as the standard unbiased or biased autocorrelation estimate methods for this task. Next, the causal image elements $r_x^+(k)$ for $0 \leq k \leq u$ are generated according to relationship (33). The c_k coefficient estimates are then determined from recursion (36) in which the a_k coefficients generated from solving (or approximating) relationship (20) are utilized. Finally, the resultant

ARMA model spectral estimate is achieved by substituting expression (38) with these computed a_k and c_k coefficients into relationship (35).

It is of interest to note that the theoretical homogeneous identity (37) can be used as a convenient measure of the ARMA model's compatability with the provided time series observations. Specifically, if the auxiliary elements as computed from expression (37) are not satisfactorily close to zero for $m > \max(q,p)$, this is indicative of an incompatability. This incompatability can be caused from such factors as an inadequate model order selection (p,q) , poor estimates of the ARMA model's a_k autoregressive coefficients, or, inaccurate estimates of the autocorrelation elements $r_x(0), r_x(1), \dots, r_x(m)$.

VII. NUMERICAL EXAMPLES

ARMA Model Example

The unmodified ARMA modeling method of spectral estimation, as presented in Section III, has been found to possess a significantly superior performance when compared to such contemporary alternate methods as the periodogram, maximum entropy, and, the Box-Jenkins method when applied to "narrow"band time series (i.e., sum of sinusoids and white noise [1], [2], [20]). With this in mind, the effectiveness of both the unmodified and modified ARMA modeling procedures will now be examined for a "moderately wide band" time series. In particular, we shall treat the time series as recently considered by Bruzzone and Kaveh [21]. Specifically, their ARMA time series of order (4,4) is characterized by

$$x_k = x_k^1 + x_k^2 + 0.5\epsilon_k \quad (39a)$$

where the individual time series x_k^1 and x_k^2 are generated according to

$$\begin{aligned} x_k^1 &= 0.4x_{k-1}^1 - 0.93x_{k-2}^1 + \epsilon_k^1 \\ x_k^2 &= -0.5x_{k-1}^2 - 0.93x_{k-2}^2 + \epsilon_k^2 \end{aligned} \quad (39b)$$

in which the ϵ_k , ϵ_k^1 , and ϵ_k^2 are uncorrelated Gaussian random variables with zero mean and unit variance. It then follows that the spectral density characterizing the time series (39) is given by

$$S_x(\omega) = \left| 1 - 0.4e^{-j\omega} + 0.93e^{-j2\omega} \right|^{-2} + \left| 1 + 0.5e^{-j\omega} + 0.93e^{-j2\omega} \right|^{-2} + 0.25 \quad (40)$$

Using the time series description (39), twenty different sampled sequences each of length 64 were generated. These twenty observation sets were then used to test various spectral estimation methods. In Figure 1, the twenty superimposed plots of the ARMA model spectral

estimates of order (4,4) obtained using the first iterate of the Box-Jenkins method, and, this paper's unmodified method with $\lambda_{kk} = (0.95)^{k-1}$ and selections of $t = 4, 8, \text{ and } 20$ are shown. For comparison purposes, the ideal spectrum (40) is also shown. From these plots, two observations may be made: (i) the unmodified method with $t = 4$ yields a marginally better spectral estimate than the Box-Jenkins method, and, (ii) the unmodified spectral estimates improves significantly as t is increased from the minimal value 4. This latter observation is most noteworthy and indicates that the incorporation of more than the minimal number of Yule-Walker equations for determining the ARMA model's autoregressive coefficients has the anticipated effect of significantly improving spectral estimation performance.

To test the effect of an improper model order selection, the same twenty observation sets were next used to obtain an ARMA model of order (6,6). The spectral density estimates which resulted, when using the Box-Jenkins, and this paper's unmodified method with $t = 6, 12, \text{ and } 30$ are shown in Figure 2. The quality of the spectral estimates has improved over that of the proper order selection (4,4) for all methods. It is noteworthy, however, that the averaged normalized determinant (32) for the unmodified methods fell by a factor of 10^{-3} as p is increased from 4 to 6 thereby properly indicating an excessive model order selection. Inexplicably, the mean condition number for the Box-Jenkins method increased when p was increased from 4 to 6.

Next, the modification methods developed in Section IV were applied to these twenty different sampled sequences of length 64 to obtain ARMA model spectral estimates of order (4,4). The resultant spectra are shown in Figure 3 where it is apparent that only "a modest" degradation in

spectral estimation performance has accrued due to the transient effects introduced by the modified methods. This is indeed welcomed news given the ability to implement these modified methods with exceptionally fast algorithms. It is to be noted that the "postmodified", and the "pre and postmodified" methods are identical in this example.

As a final example, twenty different sampled sequences each of length 200 were generated according to expression (39). With this longer data length, it was anticipated that an improvement in spectral estimation performance would result. A marked improvement is in fact realized as is made evident from Figure 4 where the ARMA model spectral estimates of order (4,4) are shown for the Box-Jenkins method and the unmodified method for selections of $t = 4, 8$, and 20.

AR Model Example

In order to further demonstrate the effectiveness of selecting t larger than the minimum value p , let us consider the time series composed of two sinusoids at normalized frequencies 0.4 and 0.5 in additive white noise such that the individual SNR's are 10dB and 0dB, respectively, that is

$$x(n) = \sqrt{10} \sin(0.4\pi n) + \sin(0.5\pi n) + \epsilon(n)$$

Twenty different samples of this time series each of length 64 were then generated. Next an AR model of order eight (i.e., $p=0$, $q=8$) using equation (21) for each of these twenty samples were generated for selections of $t=8$, and, $t=24$. The resultant spectral estimates are shown in Figure 5 where it is clear that the standard covariance method (i.e., $t=8$) cannot resolve the two sinusoids while the AR model for which $t=24$ easily achieves the desired resolution. This is a most noteworthy observation in that most AR modeling techniques (e.g., the covariance, the maximum entropy, and, many fast LMA methods) either implicitly or explicitly specify $t=q$ and thereby have an inherent inferior spectral estimation performance basis.

VIII. CONCLUSION

A fast algorithmic method for achieving an ARMA model of a stationary random time series has been presented. This method's development is predicated on the concept of approximating the underlying model's Yule-Walker equations through specially structured data dependent matrices and vectors. The chosen structures enables one to solve for the ARMA model's autoregressive coefficients using an exceptionally efficient algorithm. In particular, the computational efficiency of this algorithm is competitive with such AR model procedures as the maximum entropy and recently developed fast LMS algorithms [11]-[13].

In addition to its fast algorithmic implementation capability, this new method possesses a superior spectral estimation performance. It was shown to be an extension of the high performance method of Cadzow [1] and [2] which has been empirically found to provide superior spectral estimates when compared to such alternatives as the maximum entropy and Box-Jenkins methods. Given its dual capability of high spectral estimation performance and fast computational implementation, this new method promises to be a primary spectral estimation tool.

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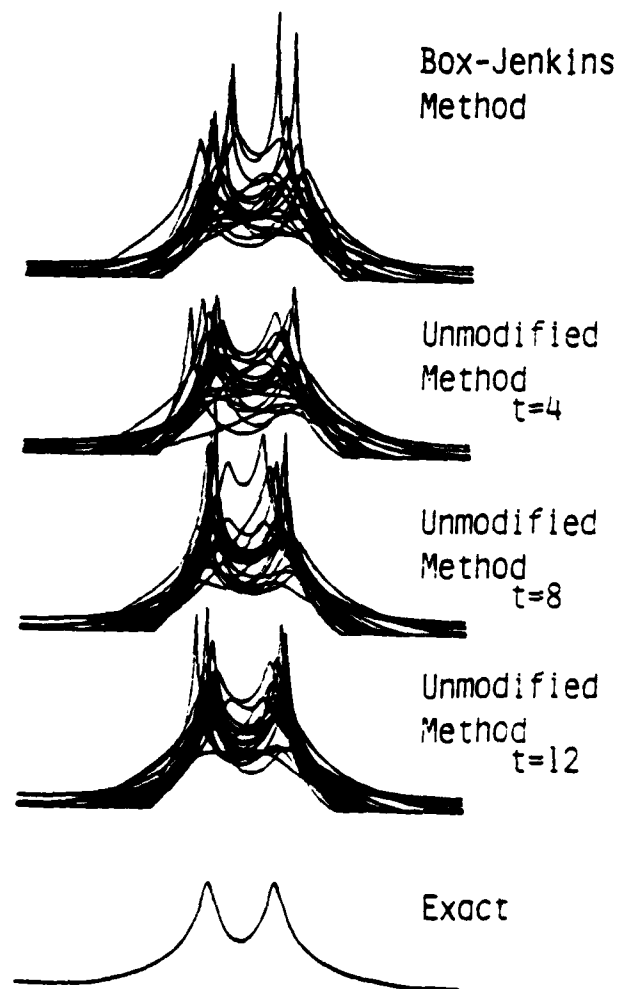


Figure 1: ARMA Spectral Estimates of Order (4,4), Data Length of 64, and, $\rho = 0.95$.

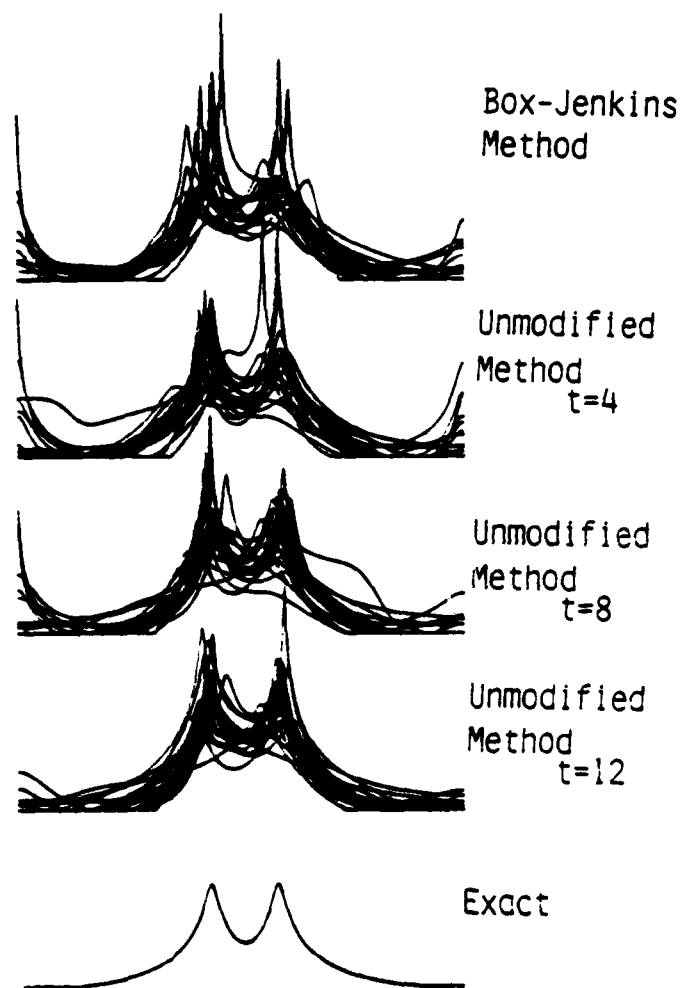


Figure 2: ARMA Spectral Estimates of Order (6,6),
Data Length of 64, and, $\lambda = 0.95$.

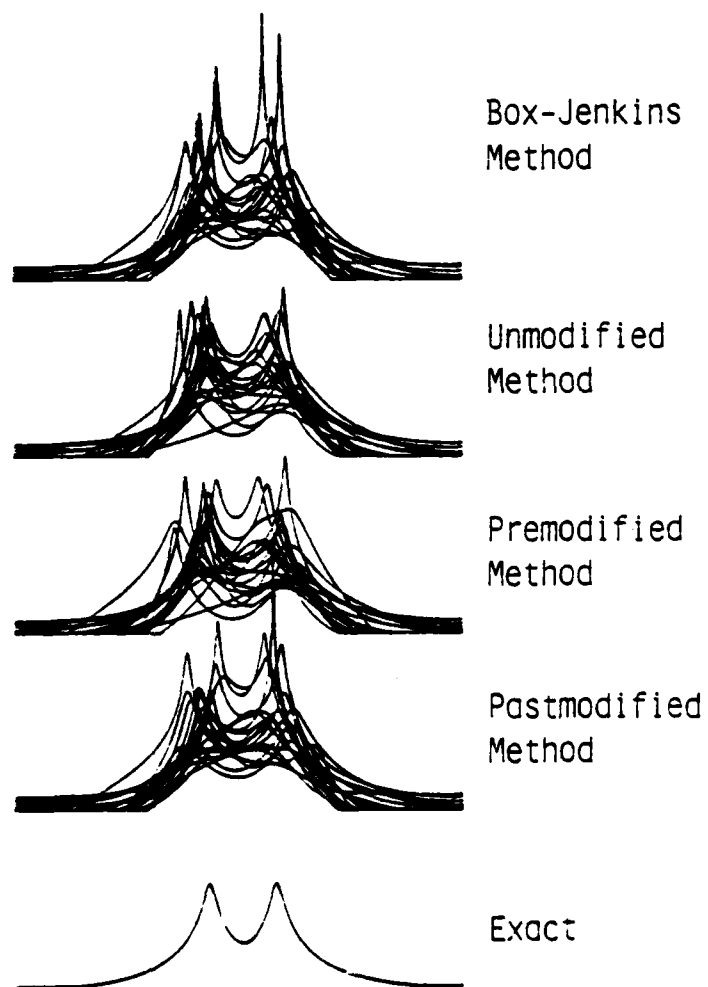


Figure 3: ARMA Spectral Estimates of Order (4,4),
Data Length 64, and, $\lambda = 0.95$.

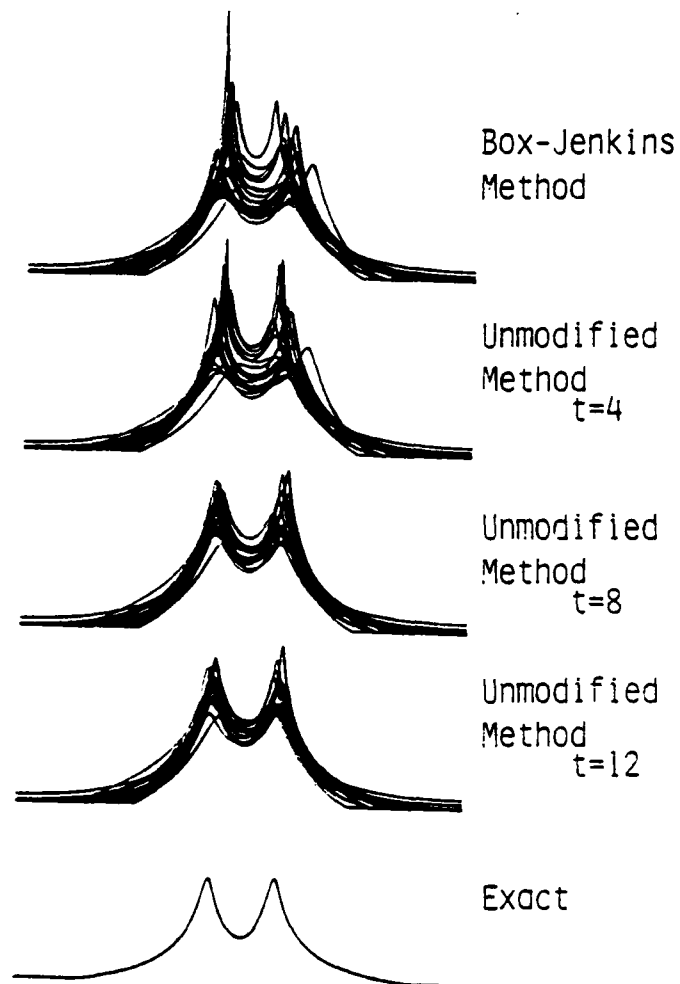


Figure 4: ARMA Spectral Estimates of Order (4,4),
Data Length of 200, and, $\lambda = 0.95$.

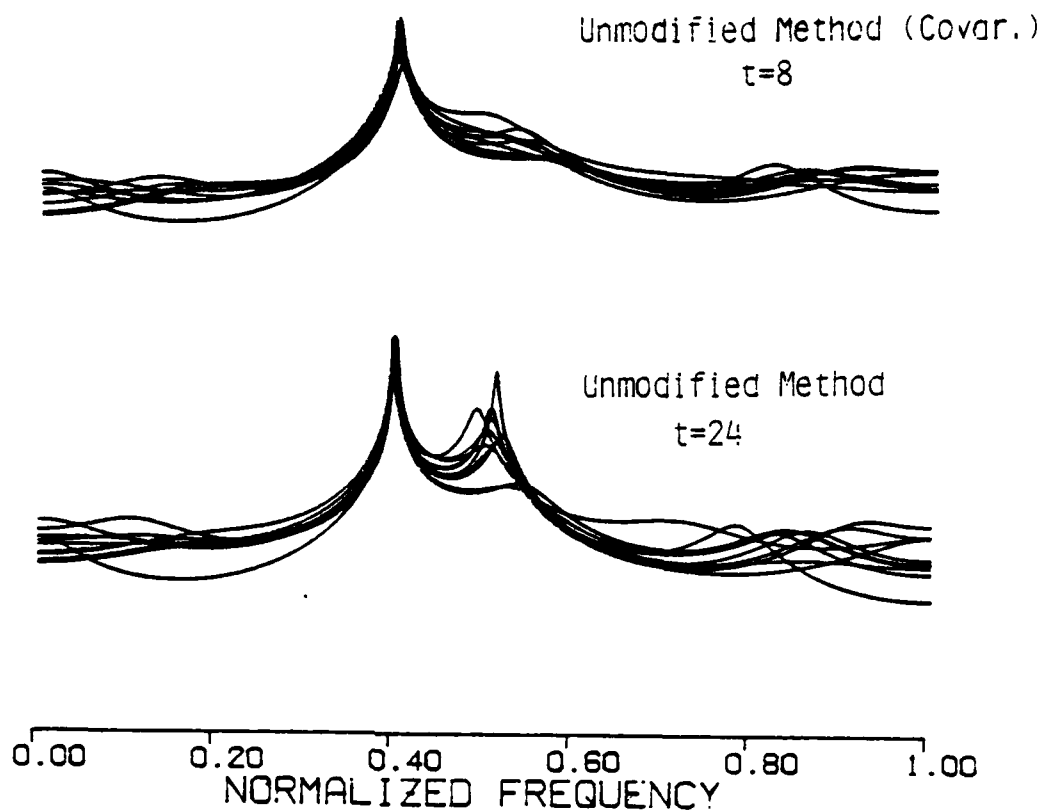


FIGURE 5: Two Sinusoids in Additive White Noise with Frequencies 0.5 (10 dB) and 0.4 (0 dB). AR Models of order 8 with a Data Length of 64.

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